Distinguished Constituents,

Semantical Analysis of Modal Logic, and

The Problem of Entailment

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definition by transfinite induction (t.i) and (d′) *proof* by t.i. *Theorem*: *Each instance of* (*d*′) *is derivable by* (*a*)–(*d*), (*c*′), when (d) is applied to all formulae in the extended notation, i.e. proof by t.i. is *reduced* to definition by t.i. . Given τ, by (c'_{ψ}) we define the descending sequence $\psi(n, 0) = n$, $\psi(n, x') = \tau^*[\psi(n, x)]$. By ($c'_{\varphi}$) we define the 'length' φ of this sequence, $\varphi(0) = 0$, $\varphi(x') = \varphi[\tau^*(x)]'$, whence $\varphi(x) = 0 \rightarrow x = 0$. By (d), $\psi(n, x) \neq 0 \rightarrow \varphi[\psi(n, x)] + x = \varphi(n)$. Putting $x = \varphi(n)$, $\psi[n, \varphi(n)] = 0 \vee \varphi\{\psi[n, \varphi(n)]\} = 0$, i.e. $\psi[n, \varphi(n)] = 0$, i.e. $\varphi(n)$ is the length of the descending part of $\tau^*(n)$, $\tau^*[\tau^*(n)]$, …, etc. Suppose $A[\tau^*(x)] \rightarrow A(x)$; then $A[\psi(n, x')] \rightarrow A[\psi(n, x)]$ and so $\{\neg A[\psi(n, 0)] \rightarrow \neg A[\psi(n, x)]\} \rightarrow \{\neg A[\psi(n, 0)] \rightarrow \neg A[\psi(n, x')]\}$, whence, by (d), $\neg A(n) \rightarrow \neg A[\psi(n, x)]$. Put $x = \varphi(n)$, and so $A(0) \rightarrow A(n)$, as required. *Application*: In the author's quantifier-free system F_1 [this JOURNAL, vol. 17 (1952), p. 47, para. 38] the schema for 'ordinal induction of finite order' can be replaced by the ordinary schema of complete induction. *Remark* 1. The parallel question of dropping (c′) in favour of (d′) is somewhat artificial because then the φ_n would not be identified. *Remark* 2. The argument can be iterated for relations $a \prec b$ in the new notation. (*Received October 25, 1959.*)

SAUL A. KRIPKE. *Distinguished constituents.*

The device of "distinguished" constituents of Ackermann XXII 327(2) can be adapted to the problem, posed in Curry's XVI 56, of finding a formulation of his LD satisfying the Gentzen subformula principle. We allow plural right sides, with some constituents distinguished; the primes have form X, A $\Vdash$A, Z^*. The positive part of the system is like LC, with the principal constituents not distinguished. For Pr (and Πr if present) we assume that all parametric constituents on the right are distinguished (the "strong" restriction). Nr is as in LK; Nl is also as in LK, but with the restriction that there must be a parametric, negated, non-distinguished constituent on the right. We adopt a rule Dr, subject to the same restrictions as Nl, allowing a non-distinguished constituent to become distinguished. A rule Dr′, allowing us to make a distinguished constituent non-distinguished at will, completes the new LD system.

If we weaken the restriction on Pr (and Πr) to assert that all parametric constituents on the right are distinguished or negated, the result is LG; HG is characterized by adding $\neg\neg(\neg\neg A \supset A)$ to HD. If we drop the notion of distinguished constituents from LD (or LG), the result is LE ("classical refutability", cf. XXII 330 (4), axioms 1–11). The positive part of LD gives a plural LA; this can be extended to plural versions of LM and LJ. Define LAV (LCV) by adding negation as a *verum* operator to LA (LC). Then LE = LCV $\cap$ LK, LG = LAV $\cap$ LK. Using distinguished constituents, we can also define LB = LAV $\cap$ LJ (HB = HM + $\neg\neg(\neg\neg A \supset A)$). All these systems have certain variant formulations using distinguished constituents, as well as singular, T, and H formulations. Throughout $A_1, \ldots, A_m \Vdash B_1, \ldots, B_n, C_1^*, \ldots, C_p^*$ can be interpreted as $A_1 \wedge \ldots A_m \,.\!\supset\!.\, B_1 \vee \ldots B_n :\!\vee\!: C_1 \vee \ldots C_p$ (in LD, equivalently: $A_1 \wedge \ldots A_m \wedge \neg C_1 \wedge \ldots \neg C_p \,.\!\supset\!.\, B_1 \vee \ldots B_n$). We obtain generalized Glivenko theorems, and, of course, elimination theorems. (*Received September 1, 1959.*)

SAUL A. KRIPKE. *Semantical analysis of modal logic.*

Semantical completeness theorems are now available for various systems of modal logic, using an appropriate model-theory to define completeness for each system, and using Beth's semantic tableaux to facilitate the proof. The systems involved are: (1) Lewis's S2, S3, S4, S5; Feys- Von Wright's M; Lemmon's E2, E3, E4, E5′ (XXIII 346); related systems intermediate between S2 and M; systems using the *Brouwersche* axiom; S6, S7, S8; various systems of deontic logic; modifications in the direction of Prior's Q. These methods lead to simple decision procedures, infinite matrices, natural deduction methods, etc., for all systems mentioned. (2) Quantifiers can be added, with completeness theorems preserved. The axiom $(x)\Box A \supset \Box(x)A$ turns out to hold when there are no "possible existents" beyond the individuals of the real world. The problems regarding necessary existence raised by Prior can be solved by several approaches, including systems like his Q, and alternatives to this. (3) If identity is added, completeness theorems can be derived either on the assumption $(x, y)\ (x = y \supset \Box x = y)$ or without this assumption. The resulting semantical notions shed new light on questions such as the morning star paradox, and provide a semantical apparatus for sense and denotation, extension and intension, and related concepts. (4) The methods for S4 yield a semantical apparatus for Heyting's system which simplifies that of Beth. They also suggest certain metamathematical applications of modal logic. (For systems based on S4, S5, and M, similar work has been done independently and at an earlier date by K. J. J. Hintikka.) (*Received October 21, 1959.*)

SAUL A. KRIPKE. *The problem of entailment.*

Consider the pure implication part of Curry's LK (as in his ***Theory of formal deducibility,*** but with K as postulated rules, and primes of form A $||\!\!-$ A only.) The resulting "material implication" is "paradoxical". We eliminate some paradox by restricting Pr to be singular on the right; this yields LJ. If we restrict LJ by abandoning the rules K, we get Church's weak implication. If we restrict Pr in LJ so that all parametric constituents are of form $C \supset D$, we get the pure implication part of S4. If we apply *both* restrictions on Pr to LJ, we get a formulation of I (defined by Belnap from (1)–(4), α and δ, in Ackermann XXII 327). For all these systems, the elimination theorem holds. The decidability theorems are problematic in the absence of K, since the usual methods depend on the presence of the converse of W; but this difficulty has been circumvented by a more general argument not requiring this rule, yielding in particular decision procedures for I and weak implication.

If we wish to add other connectives (conjunction, disjunction, negation, quantification) various alternative sets of rules can be used, which, although equivalent in the presence of K, are not equivalent in its absence.

The rule K was dropped because it allowed the introduction of "irrelevant" constituents. If we are interested in a "minimal logic" (Church), we might consider dropping the rules W, or placing even stronger restrictions on Pr. (My thanks to A. R. Anderson and Nuel D. Belnap for stimulating my interest in these problems.) (*Received October 21, 1959.*)

W. V. QUINE. *Eliminating variables without applying functions to functions.*

Schönfinkel's elimination of variables used functions which applied to themselves and one another. A general set-theoretic ontology seems called for to house these objects. I shall show, in contrast, how to eliminate variables by adopting six functors which operate iteratively on the primitive predicates, whatever they may be, to yield predicates defined over the original universe alone. The functors are: (1) *Complementation.* Applied to an n-place predicate, it gives the complementary n-place predicate. (2) *Cartesian multiplication.* Applied to an m-place predicate and an n-place predicate, it gives the $(m + n)$-place predicate which the name suggests. (3) *Extreme permutation.* Applied to a predicate of $n > 1$ places, it gives a predicate satisfied by the n-tuples which we get from those satisfying the original predicate when we permute their initial places to final position. (4) *Penultimate permutation.* Similar, but with penultimate places permuted to initial position. (5) *Fusion.* Applied to a predicate of $n + 1$ places, it gives a predicate satisfied by those n-tuples which, with their last place repeated, satisfy the original predicate. (6) *Projection.* Applied to a predicate of $n + 1$ places, it gives a predicate satisfied by the n-tuples obtainable