Transfinite Recursions on Admissible Ordinals, I,

Transfinite Recursions on Admissible Ordinals, II, and

Admissible Ordinals and the Analytic Hierarchy

Saul A. Kripke

These are published abstracts from Smullyan, R. M. (1964), Meeting of the Association for Symbolic Logic. *Journal of Symbolic Logic*, 29(3): 150-162, which can be obtained from the publisher at https://doi.org/10.2307/2271646.

C. E. M. YATES. *On the degrees of index sets.*

Let $R_0, R_1, \ldots,$ be an enumeration of the r.e. sets under a standard indexing. Let a be an arbitrary r.e. degree and $\Sigma_3(a)$ the class of all sets S for which there is a predicate Γ recursive in a such that $x \in S \equiv (\exists u)(v)(\exists w)\Gamma(x, u, v, w)$. *Def.* $e \in G(a)$ if and only if R_e is of degree a. An exact classification of $G(a)$ is given by the following theorem. *Theorem 1.* $S \in \Sigma_3(a) \equiv S \leqq_{1-1} G(a)$. In particular this solves a problem of Hartley Rogers by giving an exact classification of $G(\mathbf{0}')$. The classification of $G(\mathbf{0})$ is already known. *Theorem 2.* If c is r.e. in $\mathbf{0}'''$ and $\mathbf{0}''' \leqq c \leqq \mathbf{0}''''$ then there is a r.e. degree a such that $G(a)$ is of degree c. The classification provides an alternative proof of Sacks' theorem that if $\mathbf{0} < b < a$ and a is a r.e. degree then there is a r.e. degree d such that $b|d$ and $d < a$. (*Received March 6, 1964.*)

CALVIN C. ELGOT and ABRAHAM ROBINSON. *Random access-stored program machines.*

A notion of "Random access-stored program machine (RASP)" is introduced in order to capture the most salient features of the central processing unit of a modern digital computer. An instruction of such a machine is understood as a mapping from states of the machine into states. It can be proved that programs of finitely determined instructions are properly more powerful if address modification is permitted than when it is forbidden. This throws some light on the role of address modification in digital computers. (*Received March 24, 1964.*)

SAUL KRIPKE. *Transfinite recursions on admissible ordinals,* I.

Let α be a (von Neumann) ordinal. Consider a language like the formalism of recursive functions in Kleene's book, pp. 263 ff., except that the successor symbol is omitted, and we have a numeral $\boldsymbol{y}$ for each ordinal $y \in \alpha$, and symbols $\exists$ and $<$. The formation rules for *terms* are like Kleene's, with the stipulation that $(\exists x < t_1)t_2$ is a term for all terms t_1, t_2 where x is a variable not occurring free in t_1. (Intuitively, it is a function which is 0 if $t_2 = 0$ for some x less than t_1, and 1 otherwise.) For deductions from a *finite* system of equations E, we use Kleene's rules R1 and R2, together with R3: (a) if $n < y < \alpha$ and $\boldsymbol{n}$ and $\boldsymbol{y}$ are corresponding numerals, infer $(\exists x < y)(t(x)) = 0$ from $t(n) = 0$ (t(x) being a term "involving" x); (b) if $t(\boldsymbol{n}) = 1$ is provable for each $n < y$, infer $(\exists x < \boldsymbol{y})(t(x)) = 1$.

Given a finite system of equations E, define sets S_x for each ordinal x, by $S_0 = E$, $S_{x+1} = E \cup$ the set of all conclusions obtained from premises in S_x by R1–R3, and for y a limit ordinal $S_y = \bigcup_{x<y} S_x$. If $S_\alpha = S_{\alpha+1}$ for every E, we say α is *admissible.*

For admissible α, we can define a function ϕ with domain α and range $\subseteq \alpha$ to be *α-recursive,* if there is a finite system of equations E with principal function letter f such that $E \vdash f(\boldsymbol{x}) = \boldsymbol{y}$ iff $\phi(x) = y$. Similarly for partial recursiveness, recursive enumerability, and other standard notions of recursion theory. (The notion of partial recursive functional has, in addition to its "correct" generalization an "incorrect" generalization which we call a "pseudo partial recursive" functional.) *The basic theorems of recursion theory in Kleene's book all go through for recursion on any admissible ordinal α.* (They also go through for recursion on the class of all ordinals.) Ordinary recursion theory is the special case $\alpha = \omega$. (*Received April 21, 1964.*)

SAUL KRIPKE. *Transfinite recursions on admissible ordinals,* II.

We continue the preceding abstract, and list *some* of the main theorems of the theory.

Theorem 1. Every infinite initial ordinal is admissible.

Theorem 2. For any ordinal $\beta > 0$, there are $\aleph_\beta$ admissible ordinals $< \Omega_\beta$ (the βth infinite initial ordinal).

Large classes of "interesting" ordinals, other than initial ordinals, turn out to be admissible. (See, *e.g.*, the following abstract.)

For admissible α, and $\beta < \alpha$, a set contained in β is called *bounded* in α. An α-recursive set bounded in α is called *α-metafinite*. (This term is borrowed from Kreisel-Sacks.)

Theorem 3. If α is admissible, a subset of α is α-metafinite iff it is constructible with order $< \alpha$.

Theorem 3 is a special case of a theorem characterizing the constructible sets with order $< \alpha$. The latter theorem is proved using a result concerning a set theory S obtained by weakening Zermelo-Fraenkel: Let M be a complete model of S. Then the least ordinal not in M is admissible. Conversely, for admissible α, the sets constructible by ordinals $< \alpha$ form a complete model of S.

α is *projectible* into β (α admissible, $\beta \leqq \alpha$) if there is a 1-1 α-recursive function with range contained in β. If $\beta < \alpha$, we say α is *projectible*. A generalization of the Friedberg-Mucnik theorem can be proved for all admissible α projectible into a regular initial ordinal, and in certain other cases. For admissible, not projectible ordinals, Myhill's creative sets theorem generalizes; for projectible admissible ordinals, all creative sets are 1-1 equivalent, but only *bounded* creative sets need be isomorphic.

Our theory should be compared with those of Takeuti and Machover for cardinals and with that of Kreisel-Sacks for Church-Kleene's ω_1. Our formalism closely resembles Machover's, Theorem 1 is based on methods of Takeuti, and the generalized Friedberg-Mucnik theorem was independently proved (for ω_1) by Sacks. (*Received April 21, 1964.*)

SAUL KRIPKE. *Admissible ordinals and the analytic hierarchy.*

We assume the preceding abstracts. Define $\omega_n =$ the least countable ordinal not Δ^1_n. We can define a Σ^1_1 (Π^1_2) well ordering $R_1(x, y)$ ($R_2(x, y)$) of order type ω_1 (ω_2) such that there is a Π^1_1 (Σ^1_2) relation $S_1(x, y)$ ($S_2(x, y)$) which coincides with $R_1(x, y)$ ($R_2(x, y)$) for every y in the field of R_1 (R_2). If $x \in \omega_1$ (ω_2), by $\pi_1(x)$ ($\pi_2(x)$) we mean the unique natural number n in the field of R_1 (R_2) which determines an initial segment of order type x (a "notation" for x).

Theorem 1. ω_1 is admissible. A set $A \subseteq \omega_1$ is ω_1-r.e. (ω_1-metafinite) iff $\pi_1(A)$ is Π^1_1 (hyperarithmetic). The function π_1 is ω_1-recursive. A set $A \subseteq \omega$ is ω_1-r.e. (recursive) iff it is Π^1_1 (hyperarithmetic).

Theorem 2. ω_2 is admissible. A set $A \subseteq \omega_2$ is ω_2-r.e. (metafinite) iff $\pi_2(A)$ is $\Sigma^1_2(\Delta^1_2)$. The function π_2 is ω_2-recursive. A set $A \subseteq \omega$ is ω_2-r.e. (recursive) iff it is $\Sigma^1_2(\Delta^1_2)$.

Given a set $B \subseteq \omega$, we can define relativized versions π^B_1 and π^B_2 of π_1 and π_2 on ω^B_1 and ω^B_2.

Theorem 3. ω^B_1 is admissible. If B is constructible and its order is $< \omega^B_1$, then a relativized version of Theorem 1 holds.

Theorem 4. ω^B_2 is admissible. If B is constructible, a relativized version of Theorem 2 holds.

In Theorem 4 (3) it can be shown that conversely, Theorem 2 (1) relativizes only if B is constructible (and its order is $< \omega^B_1$). There exist sets B which are constructible but have order $> \omega^B_1$.

Analogues to Theorems 2 and 4 hold if we either assume $V = L$, or (better), consider the *constructible analytic hierarchy* (where all function quantifiers range over *constructible* number-theoretic functions).

Theorem 5. Let σ be the least ordinal which is not constructible analytic. Then σ is admissible. The σ-recursive subsets of ω are precisely the constructibly analytic sets.

For a constructible set $A \subseteq \omega$, a relativized version of Theorem 5 holds. (*Received April 21, 1964.*)