A Theory of Truth, I. Preliminary Report, and

A Theory of Truth, II

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given operations on $\mathscr{F}$ are monotonic with respect to $\leq$, (ii) $\forall\varphi\forall\psi(\forall x(\varphi x \leq \psi x) \to \varphi \leq \psi)$ and (iii) each chain $\mathfrak{M}$ in $\mathscr{F}$ has an upper bound ψ satisfying the condition

$$\forall\varphi\forall x\forall\chi(\forall\theta(\theta \in \mathfrak{M} \to \varphi\theta x \leq \chi) \to \varphi\psi x \leq \chi).$$

The iteration of φ controlled by χ is by definition the least solution θ of the equation $\theta = (\chi \supset I, \theta\varphi)$. A "function" ψ is called recursive in some "functions" $\psi_1, \cdots, \psi_n$ iff ψ can be obtained from $I, L, R, T, F, \psi_1, \cdots, \psi_n$ by means of the three given operations on $\mathscr{F}$ and iteration. For this notion of recursiveness we prove a normal form theorem, an enumeration theorem and the first and second recursion theorems.

Consider a set M together with a pairing mechanism J, L, R on it and let $c_i \in M_i \subseteq M$ $(i = 0, 1)$, $M_0 \cap M_1 = \varnothing$ (using the notations from [1], we can for example take M to be the set B^* corresponding to an arbitrary set B and take $J = \lambda st.(s, t)$, $L = \pi$, $R = \delta$, $M_0 = B^0$, $M_1 = B^* - B^0$, $c_0 = 0$, $c_1 = 1$). We obtain a model for the given system of axioms taking $\mathscr{F}$ to be the set of all partial multiple-valued mappings of M into M with the natural rule of composition and the natural partial ordering and taking $\mathscr{C} = \{\lambda s.c : c \in M\}$, $(\varphi, \psi) = \lambda s.J(\varphi(s), \psi(s))$, $(\chi \supset \varphi, \psi) = \lambda s.\{t : (\chi(s) \cap M_0 \neq \varnothing \wedge t \in \varphi(s)) \vee (\chi(s) \cap M_1 \neq \varnothing \wedge t \in \psi(s))\}$; $T = \lambda s.c_0$, $F = \lambda s.c_1$; in the special case which corresponds to [1] our notion of recursiveness will be equivalent to absolute prime computability. Other models can be obtained by taking $\mathscr{F}$ to be the set of the partial mappings of M into M, or by considering fuzzy or probabilistic mappings of M into M. We can also take $\mathscr{F}$ to be the set of all pairs $\langle D, f\rangle$, where f is a partial multiple-valued mapping of M into M and $D \subseteq \operatorname{Dom} f$, and define multiplication by $\langle D_0, f_0\rangle \cdot \langle D_1, f_1\rangle = \langle D, f_0 f_1\rangle$, where $D = \{t : t \in D_1 \wedge f_1(t) \subseteq D_0\}$. Then for a suitable interpretation of the rest of the primitive notions we again obtain a model for the considered system.

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[1] Y. N. Moschovakis, *Abstract first order computability*. I, ***Transactions of the American Mathematical Society,*** vol. 138 (1969), pp. 427–464.

Saul Kripke, *A theory of truth*. I. *Preliminary report.*

Let L_0 be an interpreted first-order language whose domain D includes the natural numbers. ('First-order' is only for definiteness; generalized quantifiers, even infinitary connectives, etc., could be included without damaging the construction.) Extend it to a language L with one additional (uninterpreted) monadic predicate $T(x)$. A *partial subset* of D, $S = (S_1, S_2)$ is a pair of disjoint subsets of D; S_1 (S_2) contains the members (nonmembers) of S, S is 'undefined' elsewhere. $S \leq S'$ means $S_1 \subseteq S_1'$ and $S_2 \subseteq S_2'$. When $T(x)$ is interpreted by a partial subset of L, formulae of L can be evaluated as true, false, or undefined by an 'appropriate' 3-valued scheme. Appropriate schemes include (the natural quantificational extension of) Kleene's 3-valued logic and van Fraassen's supervaluations. (There are others; the 'weak' 3-valued logic is 'appropriate' but is an unhappy choice unless L_0 is enriched by a restricted universal quantifier.) One such 3-valued scheme is assumed to be chosen throughout.

If $T(x)$ is interpreted by $S = (S_1, S_2)$, let $\phi(S) = S' = (S_1', S_2')$, where S_1' (S_2') is the set of Gödel numbers of true (false) sentences of L under the suggested interpretation. Then if $S_1 \leq S_2$, $\phi(S_1) \leq \phi(S_2)$.

An easy theorem: ϕ *has a* (*least*) *fixed point*. Indeed, if we define $S_0 =$ the completely undefined partial set, $S_{\alpha+1} = \phi(S_\alpha)$, for β a limit ordinal $\phi(S_\beta) = \text{l.u.b. } S_\alpha$ $(\alpha < \beta)$, then the least α ('α_0') such that $S_\alpha = S_{\alpha+1}$ gives the least fixed point of ϕ. Every f.p. is $\leq$ a maximal f.p. Usually, there is no greatest f.p., but there is a greatest 'intrinsic' f.p. (f.p. compatible with every other f.p.). The intrinsic f.p.'s form a complete lattice under $\leq$.

If $T(x)$ *is interpreted by a fixed point of* ϕ, L *becomes a language with its own truth predicate.*

Saul Kripke, *A theory of truth*. II.

We continue the preceding abstract.

It is suggested that interpretations of L using fixed points of ϕ form an approximate model for the intuitive concept of (expressing a) truth in natural language. The least fixed point probably is the most natural, but the others are useful to make certain intuitive distinctions precise. ('I am false' lacks truth-value in *all* f.p.'s; 'I am true' has no truth-value in any intrinsic f.p., but has one in every maximal f.p.; etc.)

In the usual Tarski 'hierarchy' approach, if Smith says, (1) "Everything Jones says is true", he must choose a 'level' for "true". If (unbeknownst to Smith) some of Jones utterances are on too high a "level", (1) may not 'cover' everything Jones says. The present proposal assigns no 'level' to "true". If (1) has a truth-value in the least f.p. (as it will in 'normal' cases), in a sense it has a 'level' (the least β such that $(1) \in S_\beta$), but the 'level' depends on the facts about what Jones says, rather than the 'truth-predicate' of (1).

Nevertheless if L_0 contains arithmetic, a Tarski hierarchy of truth-predicates and meta-languages for L_0 can be constructed within L. The idea: If $A(x)$ is true of the formulae of L_0, $T_0(x) = T(x) \wedge A_0(x)$, L_1 is the sublanguage formed by adjoining $T_0(x)$ to L_0, etc. The construction can be continued through high ordinals $< \alpha_0$; we omit details, which are related to the hyperarithmetic hierarchy, and can be used to construct it. Formulae such as (1), however, appear on no level of the Tarski hierarchy.

One can make a fixed point two-valued by declaring $T(x)$ false wherever it was undefined. Tarski's convention T then becomes: $T(\ulcorner A \urcorner) \vee T(\ulcorner \sim A \urcorner) . \supset . (A \equiv T(\ulcorner A \urcorner) \wedge (\sim A \equiv T(\ulcorner \sim A \urcorner))$. This leads to a simple axiomatic theory of truth.

If D contains (codes of) all finite sequences of elements of D, an analogous construction allows L_0 to be extended to a language with its own satisfaction predicate. (Here the ordinal of the induction may be uncountable.) Extensions to intensional languages are also possible. Interesting technical questions and theorems arise in the investigation.

CARLOS A. INFANTOZZI, *Some results in inferential calculus.*

In this paper it is shown: if a $(\mathscr{T})$-Johansson's field $\mathscr{T}Jf$ has its operator N such that it verifies the converse antitone property, or it is bijective and Ntc holds, or it is an involution, then $\mathscr{T}Jf$ is a Boolean algebra. One arrives at the same result if the n of the $\mathscr{T}Jf$ fulfills the axiom 0 and N has the property Ntc, or if $a^* \cup b = b:a$ or $(b:a)^* = a \cap b^*$ (notations and terminology of the paper ***Introducción a la lógica algébrica**, IES* "*Secc. Mat.*", serie X, nº 26).

If only Ntc holds, then $\mathscr{T}Jf$ is a $(\mathscr{T})$-Lewis algebra. If the axiom 0 only is fulfilled in $\mathscr{T}Jf$, then this structure is a brouwerian algebra.

The paper contains applications to $(\mathscr{S})$-fields, Heyting's and Langford's algebras, Stone lattices, bi-algebras, etc.

Other characterizations of these algebras and fields are obtained in this article.

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HARVEY FRIEDMAN[1], *Systems of second order arithmetic with restricted induction.* I.

In [3], we presented results about systems of second order arithmetic based on the full induction scheme. In this abstract, we present corresponding results for systems with induction restricted to atomic formulae.

The proper language $\mathscr{L}_0$ to use for this purpose has variables n_k over ω; variables f_k^1, f_k^2, f_k^3 over 1-ary, 2-ary, 3-ary functions on ω; the number constant 0; and the function constant N

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