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ON THE APPLICATION OF BOOLEAN-VALUED MODELS TO SOLUTIONS OF PROBLEMS IN BOOLEAN ALGEBRA

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In this paper we will illustrate the application of Cohen's method to the theory of complete Boolean algebras by proving two new theorems in that theory. The second theorem answers a question of Sikorski. We will begin the paper with some elementary remarks on the relation of Boolean-valued models to forcing.

Given any Cohen forcing construction over a model $\mathfrak{M}$ of ZF, one can form a Lindenbaum algebra of statements over the resulting model $\mathfrak{N}$ by saying that two statements φ and ψ are equivalent when the empty condition (equivalently, when every condition) weakly forces $\varphi \equiv \psi$; the resulting equivalence class is called $\|\varphi\|$. The Lindenbaum algebra thus formed is an $\mathfrak{M}$-complete Boolean algebra B: If $S \in \mathfrak{M}$, $S \leq B$, then S has a supremum in B. That is, given any set $[\varphi_i : i \in I]$ of statements over $\mathfrak{N}$ which is a set in $\mathfrak{M}$, the ranks of the statements (or of some equivalent ones) can be bounded and we can form a statement φ saying that at least one of the φ_i is true; this will be the desired supremum.

If we are forcing over the universe V, then of course the model $\mathfrak{N}$ will not exist; but we can still form the Lindenbaum algebra, which will now be complete in the absolute sense.

Given any Boolean algebra B in a model $\mathfrak{M}$, we can introduce a generic $\mathfrak{M}$-complete ultrafilter $\mathcal{F}$ on $\mathfrak{M}$. The forcing conditions will simply be statements $\underline{b} \in \mathfrak{F}$, where $\underline{b}$ is a nonzero element of the Boolean algebra. $\underline{c} \in \mathfrak{F}$ extends $\underline{b} \in \mathfrak{F}$ iff $\underline{b} \geq \underline{c}$ and $\underline{b} \in \mathfrak{F}$ forces $\underline{d} \in \mathfrak{F}$ iff $\underline{b} \leq \underline{d}$. It is not hard to verify that $\mathfrak{F}$ is an $\mathfrak{M}$-complete ultrafilter:

(1) $\underline{b} \in \mathfrak{F} \Longleftrightarrow$ some condition $\underline{c} \in \mathfrak{F}$ in the generic sequence forces $\underline{b} \in \mathfrak{F} \Longleftrightarrow$ no extension forces $\underline{b} \in \mathfrak{F} \Longleftrightarrow \sim \exists \underline{d}$ s.t. $\underline{d} \leq \underline{b}$, $\underline{d} \leq \underline{c}$, $\underline{d} \neq 0 \Longleftrightarrow$ $\underline{c} \leq -\underline{b} \Longleftrightarrow \underline{c} \in \mathfrak{F}$ forces $-\underline{b} \in \mathfrak{F}$.

(2) By $\mathfrak{M}$-completeness, we mean: if $[\underline{b}_i : i \in I] \in \mathfrak{M}$, each $\underline{b}_i \in \mathfrak{F}$, and the $\underline{b}_i$ have an infimum $\underline{b}$, then $\underline{b} \in \mathfrak{F}$. Since each $\underline{b}_i \in \mathfrak{F}$, some condition $\underline{c} \in \mathfrak{F}$ forces $(\forall i \in I)(\underline{b}_i \in \mathfrak{F})$. So for each $\underline{i}$, $\underline{c} \in \mathfrak{F}$ weakly forces $\underline{b}_i \in \mathfrak{F}$. If $\underline{c} \in \mathfrak{F}$ did not force $\underline{b} \in \mathfrak{F}$, then $\underline{c} \wedge -\underline{b} \neq 0$; but $\underline{c} \wedge -\underline{b} = \underline{c} \wedge \bigvee_i -\underline{b}_i = \bigvee_i \underline{c} \wedge -\underline{b}_i$, so for some $\underline{i}$, $\underline{c} \wedge -\underline{b}_i \neq 0$. $\underline{c} \wedge -\underline{b}_i \in \mathfrak{F}$ then extends $\underline{c} \in \mathfrak{F}$ and forces $\underline{b}_i \notin \mathfrak{F}$, contrary to the fact that $\underline{c} \in \mathfrak{F}$ weakly forces $\underline{b}_i \in \mathfrak{F}$.

(3) $\underline{b} \in \mathfrak{F}$, $\underline{b} \leq \underline{c} \Rightarrow \underline{c} \in \mathfrak{F}$ -- obvious.

Taking $\mathfrak{M}$ to be the <u>universe</u> V, let B' be the Lindenbaum algebra of statements $\|\varphi\|$ as above. (The empty condition is $1 \in \mathfrak{F}$.) We map B into B' : $b \to \|\underline{b} \in \mathfrak{F}\|$. It is easy to verify that this map is a complete homomorphism, since $\mathfrak{F}$ is a generic complete ultrafilter. Since $-\underline{b} \in \mathfrak{F}$ is forced to be equivalent to $\underline{b} \notin \mathfrak{F}$, $-\|\underline{b} \in \mathfrak{F}\| \leq \|\underline{b} \notin \mathfrak{F}\| = \|-\underline{b} \in \mathfrak{F}\|$. Similarly, if $[\underline{b}_i : i \in I]$ has an <u>infimum</u> in $\mathfrak{F}$, it is preserved: $\bigwedge_i \|\underline{b}_i \in \mathfrak{F}\| = \|\forall i\ \underline{b}_i \in \mathfrak{F}\| = \|\bigwedge_i \underline{b}_i \in \mathfrak{F}\|$. If B is complete, B and B' are isomorphic; otherwise, B' is a completion of the Boolean algebra B. (B is completely embedded in B', and B' is generated by the image of B.)

These facts are well known to experts on forcing and Boolean-valued models. We can use these methods to obtain some new results.

Let $\aleph_\alpha$ be any cardinal, and introduce a Lévy generic "collapsing" function $\aleph_\alpha \to \aleph_0$ in the usual manner. The relevant type of forcing condition will be explained below.

If we force over the universe B, Solovay has shown that the resulting Lindenbaum algebra is a countably generated complete Boolean algebra and has used this to get a new proof of a theorem of Gaifman and Hales: there are countably generated complete Boolean algebras of arbitrarily high cardinality. (BAMS, 1966, pp. 282-4; Solovay presents his proof without using forcing.) Solovay's methods can be extended to obtain:

THEOREM. Every Boolean algebra B can be completely embedded in a countably generated complete Boolean algebra.

The proof uses the following result of Rasiowa and Sikorski (Fund. Math., 1950, 193-200). Let S be the family of all nonempty subsets of B which have an infimum, let $K \subseteq S$, $\overline{\overline{K}} = \aleph_0$, and let $\underline{b} \neq 0 \in B$. Then there exists an ultrafilter $\mathfrak{F}$ such that $\underline{b} \in \mathfrak{F}$ and which is complete for infima of sets in K: if $\underline{s} \in K$, $\underline{c}$ is the infimum of the elements of $\underline{s}$, and $\underline{s} \leq \mathfrak{F}$, then $\underline{c} \in \mathfrak{F}$.

Let $\underline{S}$ be as in the paragraph above, and let $\aleph_\alpha$ be the cardinality of $\underline{S}$. Using forcing on the universe V, introduce a generic collapse of $\aleph_\alpha$ onto $\aleph_0$ in a "fantasy universe" (or Boolean-valued model) V'. The forcing conditions here can be finite sequences

$\underline{x}_0, \ldots, \underline{x}_n$ of ordinals $< \aleph_\alpha$; the condition $\underline{x}_0, \ldots, \underline{x}_n$ forces the equations $f(i) = \underline{x}_i$ $(i = 0, \ldots, \underline{n})$, where f is a generic function mapping $\aleph_0$ onto $\aleph_\alpha$. Solovay has shown that the resulting Lindenbaum algebra B' is a countably generated, complete Boolean algebra. (See the notes on Scott's talks.)

Reasoning in the "fantasy universe" V', since there is a 1 - 1 function g mapping $\aleph_\alpha$ onto $\underline{S}$, in the "fantasy universe" V' we can compose g with f, and get a function h mapping $\aleph_0$ onto $\underline{S}$. Let $\underline{\tau}$ be the object in V' which is equal to the infimum in B' of the elements of $h(0)$ if this is not 0 and is 1 (the unit element of B) otherwise. (Recall that $h(0) \in \underline{S}$ and thus $h(0)$ is a subset of B with an infimum in B.) Applying the Rasiowa-Sikorski theorem inside V', since $\underline{S}$ is countable in V', $\underline{S}$ contains a filter $\mathfrak{F}$ such that $\tau \in \mathfrak{F}$ and $\mathfrak{F}$ is complete for all infima in $\underline{S}$. Let φ map each element $\underline{b}$ of $\underline{B}$ into $\|\underline{b} \in \mathfrak{F}\|$. Then as in the previous argument, since $\mathfrak{F}$ is an ultrafilter complete for infima in $\underline{S}$, the map φ is a complete homomorphism of $\underline{B}$ into $\underline{B}'$. To show that φ is an embedding, let $\underline{b} \in \underline{B}$, $\underline{b} \neq 0$. We show $\varphi(\underline{b}) \neq 0$. Let σ be the ordinal $< \aleph_\alpha$ which corresponds to $\{\underline{b}\}$ (obviously by $\{\underline{b}\} \in \underline{S}$) in the 1 - 1 correspondence between $\aleph_\alpha$ and $\underline{S}$. Since any condition that forces $f(0) = \sigma$ forces $h(0) = \{\underline{b}\}$, $\|h(0)=\{\underline{b}\}\|$ is not zero. But if $h(0) = \{\underline{b}\}$, $\tau = \inf \{\underline{b}\} = \underline{b}$. So the $\|\tau = \underline{b}\|$ is not 0. Since $\tau \in \mathfrak{F}$ is forced by the empty condition, $\|\underline{b} \in \mathfrak{F}\|$ is not zero. Q. E. D.

An alternative proof, based on the same ideas, but avoiding any mention of forcing or Boolean-valued models, will appear shortly in Fundamenta Mathematicae.

The second theorem we shall prove solves a problem of Sikorski. Sikorski defines a complete Boolean algebra B to be homogeneous of whenever φ is a complete surjection of B onto a Boolean algebra B', B and B' are isomorphic. An alternative characterization of homogeneity is the following property: For any two elements a and b of B, $a \neq 0$, $a \neq 1$, $b \neq 0$, $b \neq 1$, there is an automorphism of B mapping a onto b. (The two definitions are equivalent except in the case of the four element Boolean algebra, which is homogeneous in the second sense but not in the first.) Sikorski, in his book on Boolean algebras, raises the following question:

Is every complete Boolean algebra isomorphic to a direct product of homogeneous complete Boolean algebras?

The complete Boolean algebras which naturally appeared were always homogeneous; an affirmative answer would have given an elegant structure theorem for complete Boolean algebras. The question was treated in a paper by R. S. Pierce (A.M.S. Symposium on Lattice Theory), who conjectured that the answer was "no", but gave certain sufficient conditions for a positive answer to hold.

We can easily show that Sikorski's problem is to be answered in the negative. In fact, we can prove the following stronger result: There is a complete Boolean algebra B such that no complete homomorphism maps B onto a complete homogeneous algebra B'. If B is any homogeneous Boolean algebra, and φ is any sentence (formula without parameters) in the language of set theory, then in the Boolean-valued model using B, φ must take either the Boolean value 0 or 1. (Otherwise, if φ took some other value $\underline{a}$, $\underline{a} \neq 0$, $\underline{a} \neq 1$, then $\underline{a}$ would be an element of the Boolean algebra invariant under all automorphisms of B, since $\underline{a}$ could be characterized as the Boolean value of φ. This would contradict homogeneity.) Hence, if $\varphi(\underline{x})$ is a formula with one free variable $\underline{x}$, the sentence ψ = "$\{\underline{n} \in \omega / \varphi(\underline{n})\}$ is standard" (see Scott's notes for "standardness") is always true in a Boolean-valued model over a homogeneous complete Boolean algebra B. For $\{\underline{n}/\varphi(\underline{n})\}$ can be defined in the two-valued model as $\{\underline{n} \in \omega : \|\varphi(\underline{n})\| = 1\}$ since for each $\underline{n}$ the $\|\varphi(\underline{n})\| = 0$ or $= 1$.

Now Easton in his thesis has shown that one can obtain models of set theory in which the $\{\underline{n} \in \omega : 2^{\aleph_n} \neq \aleph_{n+1}\}$ is anything desired; in particular, it can be a Cohen generic nonconstructible set. Thus, if ψ is the statement "$\{n : 2^{\aleph_n} = \aleph_{n+1}\}$ is standard", ψ is forced by the empty condition to be false in Easton's construction. (Actually, Easton's construction proceeds in two stages: one first enlarges the ground model $\mathfrak{M}$ to a larger model $\mathfrak{N}$ by introducing a Cohen generic set A ω, and then enlarges $\mathfrak{N}$ to a model $\mathfrak{N}'$ in which $A = \{n : 2^{\aleph_n} = \aleph_{n+1}\}$. These two forcing constructions can be collapsed into one; for the details of how to do this sort of thing, see McAloon's lecture. One can alternatively describe the Boolean algebra associated with a two stage forcing construction as the Lindenbaum algebra under the equivalence relation $0 \Vdash_1 (0 \Vdash_2 \varphi \equiv \psi)$, where φ and ψ are the statements in $\mathfrak{N}'$, $\Vdash_1$ and $\Vdash_2$ are the weak forcing relations of the two successive stages of the construction.) So, in the Boolean algebra B associated with Easton's construction, the sentence ψ : "$\{n : 2^{\aleph_n} = \aleph_{n+1}\}$ is standard" has Boolean value 0. If B is a complete homomorphism mapping B onto B', then also in the Boolean value over B', ψ must take the Boolean value 0. (See Scott's lectures again.) But we showed above that if B' were homogeneous, ψ would have Boolean value 1 in B'. So B' cannot be homogeneous and B is the required Boolean algebra. Q. E. D.

After the present writer communicated this result and its proof to Solovay, Solovay showed that if one replaces Easton's construction by the more complex one of McAloon one can obtain the following extension: there is a rigid complete Boolean algebra B (i.e., a complete Boolean algebra with no automorphisms other than the identity) with more than two elements.