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TRANSFINITE RECURSION, CONSTRUCTIBLE SETS, AND ANALOGUES OF CARDINALS

Saul Kripke

Part of this article was written up by A. R. D. Mathias

We are interested in generalizing classical recursion theory (on ω) to recursion on the ordinals less than some 'admissible' ordinal α. ('Admissible' will be defined later.) Special cases of the present theory were discovered by Takeuti, Machover, and Kreisel (building on work by Mostowski _et al._); their coworkers were Kino, Lévy and Sacks respectively. The present basic theory was independently rediscovered by Platek. We do not claim this theory to be the most general type of recursion - only that it is a 'solid' and well established generalization.

Let α be any ordinal. ('Cardinal' will always mean _infinite_ cardinal; 'ordinal' will mean ordinal, finite, or infinite). We can consider a being who is given all the ordinals $< \alpha$, in their natural order, as primitive objects. For us, the natural numbers are given as surveyable objects; computations are well-ordered finite $[< \omega]$ sequences of operations. (So for us, $\alpha = \omega$.) The being we have in mind can perform any well-ordered sequence of operations of order-type $< \alpha$ in a finite time. In particular, and crucially, the being, given any ordinal $n < \alpha$, can survey the ordinals $< n$ to see if a given property holds on some ordinal $< n$

or fails on all of them. (Thus, if $\alpha > \omega$, he can decide on the Goldbach conjecture.) For him such a survey is 'finite'. At the moment, we put no restrictions on what α may be; 27 and the first measurable cardinal are equally good.

To formalize this, we set up an equation calculus as in Kleene. We have a denumerable list of function letters, $f,g,h,\ldots$, parentheses, comma, =, a denumerable list of number variables $x_1,x_2,\ldots,y$, and for each ordinal $n < \alpha$, a numeral $\bar{n}$. Variables and numerals are terms; if $t_1,\ldots,t_n$ are terms, so is $f(t_1,\ldots,t_n)$, where f is any function letter. To capture the superbeing's ability to search through a proper initial segment of α and to decide whether there is any x in that segment such that a certain property holds, we introduce some new notation. If t_1 and t_2 are terms, we say $(\exists x < t_1)t_2$ is a term, provided t_1 does not contain x free. Intuitively, t_2 will be a characteristic function of x, $t_2(x)$, t_1 will denote some ordinal $n < \alpha$, and $(\exists x < t_1)t_2$ will denote 0 if for some $m < n$, $t_2(m) = 0$, and will denote 1 if for all $m < n$, $t_2(m) = 1$. Note that $(\exists x < t_1)t_2$ is a <u>term.</u> The only <u>formulae</u> in the equation calculus have the form $t_1 = t_2$, t_1,t_2 terms. The bounded existential quantifier applies to terms, not to equations; if this confuses, think of it as a bounded infinite product.

The language we have set up depending on α will be called $\mathcal{L}(\alpha)$. If E is a finite system of equations, and $t_1 = t_2$ is an equation in $\mathcal{L}(\alpha)$, $E\vdash_\alpha t_1 = t_2$ iff $t_1 = t_2$ follows by one of the following rules. (Subscript α is often dropped if fixed by the context):

R_1) $$t_1(x) = t_2(x) \ / \ t_1(\bar{n}) = t_2(\bar{n})$$

R_2) $$t_1 = t_2, t_3 = \bar{n} \ / \ t_1^* = t_2^* ,$$

where t_1^*, t_2^* come from t_1, t_2 by replacing t_3 by $\bar{n}$.

R_3 a) $$m < n < \alpha, \quad t(\bar{m}) = 0 \ / \ [(\exists x < \bar{n})\, t(x)] = 0$$

b) if $n < \alpha$ and for all $m < n$ we have $t(\bar{m}) = 1$, conclude

$$[(\exists x < \bar{n})\, t(x)] = 1 .$$

For a fixed E, define $S_0 = E$, $S_{x+1} = S_x \cup$ all conclusions of $R_1 - R_3$ with premises in S_x, $S_y = \bigcup_{x<y} S_x$ for y a limit ordinal. $E \vdash_\alpha t_1 = t_2$ iff $(\exists x)(t_1 = t_2 \in S_x)$.

If φ is a function, whose domain is α and with range $\subseteq \alpha$, we say φ is α-recursive if there is a finite system of equations E and a letter f such that $E \vdash_\alpha f(\bar{m}) = \bar{n}$ iff $\varphi(m) = n$. Similarly for α-partial-recursive, α-r.e. set, etc.

So far, we have put no condition on α. But in our imaginary conception of a superbeing, S_x gives a certain measure of the number of steps required to deduce an equation. Clearly we want the superbeing to be able to deduce anything he wants in $< \alpha$ steps. So we say:

DEFINITION. α is <u>admissible</u> iff for any E, $S_\alpha = S_{\alpha+1}$.

This says that S_α is closed under the rules. It readily follows from admissibility that α is a limit ordinal and hence that $S_\alpha = \bigcup_{x<\alpha} S_x$. Thus the definition says that any equation that can be derived from E at all can be derived in $<\alpha$ steps, and demands this for any finite system of equations E.

Another concept is:

DEFINITION. α is <u>recursively regular</u> iff any α-partial-recursive function whose domain is a proper initial segment of α has a range bounded in α.

This concept is not such an obvious intuitively necessary condition for α to satisfy. But we have:

THEOREM. α is admissible iff α is recursively regular.

The basic fact now is: <u>All standard theorems of elementary recursion theory hold for all admissible</u> α. To prove this, we must code finite sequences of ordinals by ordinals and then Gödel number the symbols of $\mathcal{L}(\alpha)$ by ordinals. The ordinal pairing function used in the Gödel monograph is adequate for this purpose.

<u>Remarks</u>. One can also, set, as it were, α = On, and do recursion on <u>all</u> ordinals. Of course, some of the definitions have to be changed, so as to avoid illegitimate uses of proper classes; but, in the

appropriate sense, On is admissible. (In fact, if considered a cardinal, it is a 'regular cardinal', and thus a fortiori is a recursively regular ordinal.)

A subset K of α is α-bounded iff $\exists\sigma < \alpha$ s.t. $K \subseteq \sigma$. K is α-metafinite if it is α-bounded and α-recursive. 'Metafinite' sets are the infinitary analogue of finite sets; they represent the sets which the superbeing can survey as a single presented object. (He knows an upper bound α for such a set, and can tell which ordinals are in or out.) We can show that the recursion theory would not change if we allowed functions to be defined from an α-metafinite or even an α-r.e. set of equations.

We define the system PZF of weak set theory set theory as follows:

Drop from ZF the axioms of infinity and power set and the schema of replacement; add the replacement schema for restricted (Σ_0 in the sense of Lévy) formulae: i.e. for each Σ_0 formula with parameters, $\mathfrak{A}$, the wff $\forall x \in z\ \exists! y \mathfrak{A}(x,y) \rightarrow (\exists w)$ (w = image of z).

The other axioms are empty set, unordered pair, extensionality, infinite union, and regularity.

THEOREM. The following are equivalent:

i) α is admissible;

ii) α is the ordinal of ($=_{Df}$ the least ordinal not in) a transitive model of PZF;

iii) $L_\alpha \models$ PZF + V = L (where L = L[0] is as defined on page III-H-3 of this volume).

Sketch of proof:

i) $\rightarrow$ ii) We show that L_α is a model of PZF. Gödelize the language of L_α by ordinals $< \alpha$. Show that under the Gödel numbering the ε-relation between two terms in the language of L_α is α-recursive; and that restricted quantifiers over sets correspond to bounded quantifiers over ordinals. Then every restricted (Σ_0 formula corresponds to an α-recursive relation under the Godel numbering, and thus Σ_0- replacement follows from recursive regularity. The other axioms are trivial.

ii $\rightarrow$ iii) Gödel's monograph can be carried out in PZF, to the extent of defining L, showing that L is a model of PZF, and proving that V = L holds in L. Model theoretically, if a transitive $\mathfrak{M}$ with ordinal α satisfies PZF, L_α satisfies PZF + V = L.

iii $\rightarrow$ i) We said above that in recursion theory on <u>all</u> the ordinals, one can prove that On is an 'admissible ordinal'. This proof can be carried out in PZF. Since admissibility is an absolute notion, if we are reasoning inside a model L_α of PZF, the proof shows that α is admissible.

THEOREM. If α is admissible, $K \subseteq \alpha$, then K is α-metafinite $\leftrightarrow$ it is in L_α.

<u>Proof</u>. $\leftarrow$ Clear, since K is α-recursive and K is obviously bounded in L_α.

$\rightarrow$ If α = On, one can show in PZF that an α-metafinite <u>class</u> is a constructible <u>set</u>. <u>For every</u> α-<u>recursive class is defined by a</u> Δ_1

formula (see the next theorem), and by restricted Aussonderungs. Since the class is bounded, it is a set. Relativizing to L_α, every α-metafinite set is in L_α.

THEOREM. Let α be admissible. $K \subseteq \alpha$ is α-recursive (α-r.e.) iff K is ordinally definable in L_α by a $\Delta_1(\Sigma_1)$ formula.

Remark. These theorems are quite independent of the definition of L; they remain true whether L_α means M_α (the α^{th} level of the ramified hierarchy of sets), or means $F''\alpha$ in the sense of the Gödel monograph.

THEOREM. Every cardinal is admissible.

Proof. For regular cardinals, trivial. For singular cardinals, we must use the 'pressing down' argument of the Gödel proof of the consistency of the GCH.

THEOREM. For $\alpha > 0$, there are $\aleph_\alpha$ admissible ordinals $< \aleph_\alpha$.

Proof. There are $\aleph_\alpha$ models of PZF $< \aleph_\alpha$, by familiar arguments.

We now give some more examples of admissible ordinals.

We recall the system of second order arithmetic defined in Shoenfield's lectures. We write the levels of this hierarchy with superscript '1', as Σ^1_1, Δ^1_1 to avoid confusion with the settheoretical hierarchy used above. Remembering that each countable ordinal can be coded as a set of integers (as on page III-H-15) we may make the following definition: $\delta_n =_{DF}$ the least ordinal not given by a Δ^1_1 code. For each $n \in \omega$, δ_n is admissible. A theorem of Spector states that $\delta_1 = \omega_1$ recursive, the Church-Kleene ω_1 - the first ordinal with no recursive code: so recursive ω_1 is admissible; and indeed we have the following notation-free characterization: it is the least admissible ordinal. The δ_1-recursive (δ -r.e.) reals are precisely the $\Delta^1_1(\pi^1_1)$ reals. The δ_2-recursive (δ_2-r.e.) reals are precisely the $\Delta^1_2(\Sigma^1_2)$ reals, but since if there is a measurable cardinal all constructible reals are Δ^1_3, the theorem warns us not to expect anything similar for the case $n = 3$. In fact for $n > 2$, the situation is quite different for different 'extreme' assumptions like $V = L$ and measurable cardinal.

Let α,β, $\alpha < \beta$, be any ordinals not necessarily admissible.

DEFINITION. α is stable w.r.t. β iff whenever E is a finite set of equations in $\mathcal{L}(\alpha)$ and $t_1 = t_2$ is in $\mathcal{L}(\alpha)$, then $E \vdash_\beta t_1 = t_2 \rightarrow E \vdash_\alpha t_1 = t_2$. (In other words, though the β-man may use deductions not available to the α-character, he obtains no further α-equations.) Now ω is stable w.r.t. no larger ordinals: for consider the computation of $j(\mu y(y = 1 + y))$ where $j(x) = 0$ for all x. The ω-man never finds a y such that $y = 1 + y$, so $j(\mu y(y = 1 + y)) = 0$ is defined. If α is stable w.r.t. all $\beta > \alpha$, we say α is stable Every cardinal $> \omega$ is stable; and using the Addison-Kondo or

Kondo-Novikov theorem, δ_2 is stable. (It is the least stable ordinal.) If α is stable w.r.t. some $\beta > \alpha$, then α is admissible. The class of stable ordinals is closed (in the order topology), whereas the class of admissible ordinals is not.

The following definitions will lead to a model theoretic characterization of stable ordinals.

DEFINITION. Let K and K' be transitive sets or classes. Write $K \prec_{\Sigma_1} K'$ ("K is an elementary submodel of K' w.r.t. Σ_1 formulae") if $K \subseteq K'$ and every Σ_1 formula with parameters in K which is true in K' is true in K.

THEOREM. i) Let β be admissible. Then α is stable w.r.t. β iff $L_\alpha \prec_{\Sigma_1} L_\beta$

ii) α is stable iff $L_\alpha \prec_{\Sigma_1} L$.

DEFINITION. i) α <u>is projectible into</u> β, in symbols $\alpha \xrightarrow{proj} \beta$, iff there is a 1 - 1 α-recursive function whose range is a subset of β. (This concept is interesting only when $\beta < \alpha$).

ii) α is <u>projectible</u> iff there is a $\beta < \alpha$ with $\alpha \xrightarrow{proj} \beta$

iii) the least β such that $\alpha \xrightarrow{proj} \beta$ is called the <u>projectum</u> of α.

If α is admissible, the projectum of α is also admissible, but is clearly not projectible. If $\alpha \xrightarrow{proj} \beta$, we can obtain a set of <u>notations</u>, each $< \beta$, for the ordinals $< \alpha$. Both δ_1 and δ_2 project into ω.

THEOREM. α is admissible but not projectible iff $\alpha = \omega$ is a limit of smaller admissible ordinals each stable w.r.t. α.

We shall see shortly that stable ordinals are in a certain sense the counterparts of large cardinals, as they exhibit, for example, Mahlo properties. First, some more definitions:

DEFINITION. Let $\langle \tau_\alpha^0 | \alpha \in On \rangle$ be the monotonic enumeration of admissible ordinals. α is <u>recursively inaccessible</u> iff $\tau_\alpha^0 = \alpha$. Let $\langle \tau_\alpha^1 | \alpha \in On \rangle$ enumerate the recursively inaccessible ordinals ... Let $\langle \tau_\alpha^\lambda | \alpha \in On \rangle$ enumerate the ordinals α satisfying $\forall \beta < \lambda (\alpha = \tau_\alpha^\beta)$. Mahlo, however, introduced a type of number much bigger than all the numbers in this series. Mahlo defined a ρ_0-number to be a regular α such that every closed and α-unbounded subset of α contained a regular cardinal. Remembering that <u>admissible</u> = <u>recursively regular</u>, we make the following DEFINITION.

α is <u>recursively-ρ_0</u> iff α is admissible and every α-recursive closed α-unbounded subset of α contains an admissible ordinal.

We can go on to define a recursively hyper-Mahlo number as an admissible ordinal α, every recursive closed-unbounded subset of which contains a recursive ρ_0-number; and so on to hyperhyper-Mahlo numbers,...

Now if α is stable w.r.t. at least one $\beta > \alpha$, then α is recursively hyper-Mahlo,...; in short, α is Mahlo of all types.

One might therefore expect stable α to be a recursive equivalent of very large cardinals. One can also prove that an admissible non-projectible ordinal $> \omega$ is Mahlo of all types.

The first recursively inaccessible ordinal is $\omega_1^{E_1}$, the least ordinal not recursive in the functional E_1 which tells you whether a relation on ω is a well-ordering.

In formal second-order arithmetic, the Π^1_2 or Δ^1_2 comprehension axiom asserts the existence of any set of integers defined by a Π^1_2 or Δ^1_2 formula. (Allowing given sets of natural numbers as parameters.) There is a close correspondence between admissible ordinals and models of various comprehension axioms. Let $\mathfrak{M}(\alpha) = P(\omega) \cap L_\alpha$. If α is admissible, $\mathfrak{M}(\alpha)$ is a model of the comprehension axiom (in fact, of a bit stronger theory). If α is a limit of admissibles, $\mathfrak{M}(\alpha) \models \Pi^1_1$ C.A. If α is admissible and a limit of admissibles (= recursively inaccessible), $\mathfrak{M}(\alpha) \models \Delta^1_2$ C.A.) (Gandy had previously shown that $\mathfrak{M}(\omega_1^{E_1}) \models \Delta^1_2$ C.A.)

If α is admissible but not projectible into ω, then $\mathfrak{M}(\alpha)$ satisfies the Π^1_2 C.A. These results have converses which I shall not state here. (I will talk on this in more detail in Amsterdam.) Since to get a β-model of the Π^1_2 C.A. one must get to a very high (non-projectible) Mahlo number, the failure of people to construct such a model from below is explained. (A β-model is a model of second order arithmetic in which the notion of a well-ordered subset of ω is absolute. It plays a role similar to that of wellfounded models in set theory. For all comprehansion axioms other than the Δ_1^1, the models we obtain are β-models.)

We now give some more definitions, and then close with some further model-theoretic observations.

DEFINITION. Let α be admissible. α is a <u>quasi-cardinal</u> iff every 1 - 1 total α-arithmetic function has α-unbounded range.

A <u>regular</u> quasicardinal is one such that every partial arithmetic function with bounded domain has bounded range.

THEOREM. i) α is a constructible cardinal iff every α-bounded constructible subset of α is α-metafinite.

ii) α is not projectible iff every α-r.e. subset of α is α-metafinite.

iii) α is a quasi-cardinal iff every α-bounded α-arithmetic subset of α is α-metafinite iff $L_\alpha \models$ PZF + full Aussonderungsschema.

iv) α is a regular quasicardinal iff $\models$ PZF + full replacement schema.

Finally, we observe that the smallest quasicardinal $> \omega$ is the ordinal Gandy and Putnam call β_0, the ordinal of the minimal β-model of analysis. In fact, if α is a quasicardinal $\mathfrak{M}(\alpha)$ is a β-model of analysis.