

Mathematical Incompleteness Results in First-Order Peano Arithmetic: A Revisionist
View of the Early History

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Abstract. In the *Handbook of Mathematical Logic*, the Paris-Harrington variant of Ramsey's theorem is celebrated as the first result of a long 'search' for a purely mathematical incompleteness result in first-order Peano arithmetic. This paper questions the existence of any such search and the status of the Paris-Harrington result as the first mathematical incompleteness result. In fact, I argue that Gentzen gave the first such result, and that it was restated by Goodstein in a number-theoretic form.

Keywords: mathematical incompleteness; independence results; first-order Peano arithmetic; Gentzen; Goodstein.

1. The Received (*Handbook*) Account¹

The received (*Handbook*) account of the subject can be given in two parts. One is the editor's note by Jon Barwise in the *Handbook of Mathematical Logic*, introducing the classic Paris-Harrington result:

¹ This paper represented my views about the history of the subject sometime after the *Handbook of Mathematical Logic*'s characterization of the significance of the Paris-Harrington result appeared, and shortly after the publication of *Kirby and Paris 1982*, when I looked up Goodstein's original paper. Nevertheless, I only drafted the first two sections of the present paper in 2007 (though I discussed it in a seminar given at Princeton University in 1998 and I remember discussing it with some logicians as well). A little has been added (at least) in footnotes at the request of one referee. I believe that what I call 'the received (*Handbook*)' account' was indeed the received account, in spite of the reactions of Kreisel and Gödel at the

Since 1931, the year Gödel's Incompleteness Theorems were published, mathematicians have been looking for a strictly mathematical example of an incompleteness in first-order Peano arithmetic, one which is mathematically simple and interesting and does not require the numerical coding of notions from logic. The first such examples were found in 1977 [...]²

The allusion is of course to *Paris and Harrington 1977*, a simplification of earlier work of Paris.

Later, other such independence results were found, the best known being those of *Kirby and Paris 1982*. One result (their second involving 'hydras') is less widely quoted than the other, though the basic points of the hydra result are closely related to their more celebrated proof of the independence of a result of Goodstein. Here, to take a common statement of the latter result, I cite the exposition in Henle's textbook (1986, pp. 45-46).

time. However, things seem to have changed for the better. In particular, *Rathjen 2015* gives an excellent discussion of Goodstein's work and its relation to Gentzen's. It is much more technically detailed than the present paper and includes illuminating archival research on *Goodstein 1944* by Jan von Plato. Rathjen's paper appeared in 2015, but grew out of a lecture given in 2012, which touched on many of Goodstein's papers. It will be discussed in section 3.

² *Barwise 1977*, p. 1133. The present writer is unaware of such a search. He considered himself more or less aware of what logicians were trying to do in the period in question. Was he isolated from the relevant mathematical community? Moreover, the doubt expressed by *Rathjen 2015* that there was such a 'holy grail' problem as the *Handbook* supposes, and the facts about Gentzen's work show that there should not have been such a search, as will be documented in the present paper.

He calls a number, say 23, *written to the 'superbase' 2*, when all numbers greater than 2 are eliminated from its base-2 expression:

$$23 = 2^4 + 2^2 + 2^1 + 2^0 = 2^{2^2} + 2^2 + 2^1 + 2^0 \text{ (or } = 2^2 + 2^{2^2} + 2 + 1)$$

Similarly, for 514:

$$514 = 2^9 + 2 = 2^{(2^{(2+1)+1})} + 2$$

In other words, to write a number to the base k is to write it as a sum of powers of k (as a polynomial in k , with all coefficients $<k$). Suppose we apply the same representation to the exponents (in the polynomial representation) and iterate the process. Then, it is said, Goodstein defined the sequences of a (now) well-known kind. Given any term of the sequence, the base involved is replaced by its successor and one is subtracted. In general, the next item in the sequence will evidently be enormously larger than its predecessor. Nevertheless, *Goodstein 1944* showed that the sequence eventually decreases, and indeed terminates at 0! (In other words, an infinite Goodstein sequence is impossible.)

To quote Henle, Goodstein's theorem,

[...] is remarkable in many ways. First, it is such a surprising statement that it is hard to believe it is true. Second, while the theorem is entirely about *finite* integers, Goodstein's proof uses *infinite* ordinals. Third, 37 years after Goodstein's proof appeared, L. Kirby and J. Paris proved that the use of infinite sets is actually *necessary*. That is, this is a theorem of arithmetic that can't be proved arithmetically, but *only* using the extra powers of set theory!³

³ *Henle 1986*, p. 45. Analogously to the *Handbook*, *Henle 1986* (p. 48) describes a long search, really unknown to the present writer, for a proof of the statement in first-order PA, and claims that, finally, Kirby and Paris proved it impossible. As far as I am personally aware, the

2. The ‘True’ History

(a) First, it is hard for the present writer to see why it was not Gerhard Gentzen who found the first ‘mathematical incompleteness’ in PA in 1936. Only if ‘logic’ includes informal set theory (and remember that Cantor, who formulated the result in question, used no formal logic) would Gentzen’s theorem – the independence of ε_0 -induction with each ordinal $< \varepsilon_0$ represented in extended Cantor normal form (i.e., to the ‘superbase’ ω) – fail to be a ‘mathematical’ independence result. Moreover, the statement is not one ‘cooked up’ in order to be proved independent, but is a natural existing result in the prior mathematical literature. (This is in contrast with the Paris-Harrington statement.)

The very same *Handbook* that pronounces that Paris and Harrington found the first mathematical incompleteness result in first-order PA states in an earlier article by Helmut Schwichtenberg (see *Barwise 1977*, pp. 868, 869) that ε_0 -induction is an ‘equivalent formulation of (*) having a clear mathematical meaning’. ((*) is a restricted reflection principle for first-order PA: $\exists x \text{Der}_S(x, \ulcorner \varphi \urcorner) \rightarrow \varphi$). Any such reflection

Goodstein paper was generally forgotten by the community of logicians until Kirby and Paris revived it. Clear evidence that this is so is given by the fact that even today many people misstate its central result, relying on a somewhat careless reading of *Kirby and Paris 1982*. This will be documented below. *Rathjen 2015* is, of course, an exception to this assertion. Goodstein actually found a statement *equivalent* to ε_0 -induction, and therefore independent in first-order PA. (More detail will be given below.)

Let me emphasize that I have cited Henle’s textbook for expository purposes only, to give a reference in print. It expresses the common understanding in the logical community of the relevant history – I do not mean to criticize a particular author.

principle must be unprovable, by Gödel's second incompleteness theorem, and thus equivalently ε_0 -induction is unprovable.)

Remark 1. Somewhere in his voluminous writings, Kreisel also, writing about the Paris-Harrington theorem and the *Handbook's* characterization, reacted that Gentzen's theorem is equally a 'mathematical' independence result in first-order PA. (Or so the present writer recalls.)⁴

Remark 2. In the oral tradition of the time, Gödel was said to have reacted, 'What could be more mathematical than a Diophantine equation?' Indeed, by the time of the *Handbook*, *Matiyasevich 1970*, building on earlier work by *Davis, Putnam, and-Robinson 1961*, had shown, in effect, that in any consistent recursively axiomatized system (in which some weak theory such as R is interpretable), we can effectively find a Diophantine equation that has no solution, but where this fact cannot be proved in the system.

The main distinction that is important here is that between a theorem about unprovability in arbitrary consistent recursively axiomatized systems containing a modest amount of arithmetic (say, theory R) and unprovability results about specific systems, such as first-order PA. The result about unprovable true Diophantine inequalities⁵ is of

⁴ Also, of course, Kreisel (anticipated in unpublished work of Gödel) developed Gentzen's work into his 'no counterexample' interpretation of PA, and also of fragments with restricted induction. For Π_2^0 statements the characterization can be given directly, and does not need a 'no counterexample' interpretation. His notion of ordinal recursion later developed into such things as the Hardy hierarchy and the Wainer hierarchy (extensions of the Grzegorzcyk hierarchy).

⁵ That is, lack of solutions, as mentioned above. Gödel may actually have had in mind even his earlier 1934 result on Diophantine equations preceded by a string of quantifiers (*Gödel 1934*, section 8, 'Diophantine equivalents of undecidable propositions'). (In a postscript added

the first kind. It gives an effective method for finding in every one of the relevant systems a Diophantine equation that has no solutions, but where this fact is unprovable in the system.

The Paris-Harrington statement – like those of Gentzen, Goodstein, and Kirby and Paris – point out specific weaknesses in first-order PA (or first-order PA with a function letter added). They are of the second kind.

In this respect, Barwise’s editorial remark is especially misleading. Gödel’s original 1931 paper was *not* about first-order PA, but about – as in its title – ‘*Principia Mathematica* and Related Systems’. (A system amounting to first-order PA is indeed mentioned, but not stressed.) In this respect, the result on Diophantine equations does the job quite well. But if one is looking at first-order PA specifically, the situation is otherwise. (The same holds if one is looking for other statements unprovable in various particular systems.)

(b) Perhaps the issue is a subjective one. For example, Fairtlough and Wainer in the *Handbook of Proof Theory*, following the usual view of the logical community, pronounced that Gentzen’s results on ε_0 -induction and corresponding results on fragments of first-order arithmetic are “‘logic” independence results’ (Buss 1998, p. 190),

in 1964 he strengthens the result.) If so, he may have thought that he himself had given a ‘mathematical’ independence result from the very beginning.

The result of Matiyasevich, however, is stronger, needing no string of other quantifiers before the universal statement of an unsatisfiable Diophantine equation. No doubt Gödel would have acknowledged that this was an improvement. For the present writer’s point of view of the situation, see the text above.

as opposed to the Paris-Harrington result, though at least they mention Gentzen in this connection so that the reader can form her own judgment.⁶

I should add that in no way do I intend to denigrate the classic and beautiful Paris-Harrington result, which has influenced an enormous amount of fruitful work,⁷ though in my opinion it should be called the first result in finite combinatorics, rather than the first ‘mathematical’ result, shown to be independent in first-order PA. However, the truth of the Paris-Harrington statement is normally proved as a consequence of the infinite Ramsey theorem. It is very hard for the present writer to see why this result, and hence its consequences, is not just as much a set-theoretic result as ε_0 -induction. In fact, I see

⁶ However, see the remarks below on the original Goodstein paper. I think no one denies that it is a number-theoretic proposition, as Goodstein says. I have already stressed (see footnote 3 above) that in its original formulation it is *equivalent* to ε_0 -induction, as Goodstein rightly emphasizes.

⁷ Actually, I confess that the Paris-Harrington statement does seem to me to have a certain defect of beauty. The original finite Ramsey theorem was proved for arbitrary finite sets. The cardinality of the set was all that mattered, not the identity of the elements. The Paris-Harrington statement required the identification of the elements with particular natural numbers.

However, in *Graham, Rothschild, and Spencer 1990*, a theorem of Van der Waerden is called in the preface (p. 5) ‘the central result of Ramsey’s theory’. The theorem states that if the positive integers are partitioned into two classes, then at least one of the classes must contain arbitrarily long arithmetic progressions (see p. 29). It is obviously about the positive integers.

Certainly, and regardless of any such considerations, the statement in the text about the Paris-Harrington having inspired a great deal of fruitful work still stands, my own work on ‘fulfillability’ included.

even the finite results as set-theoretic, and thus properly included in the *Handbook of Mathematical Logic* in that section, even though they would have been intelligible, and might have been discovered, before set theory became a common tool of mathematics. Of course, ε_0 -induction is equivalent to the statement that there are no infinite descending chains (in extended Cantor normal form) of ordinals $<\varepsilon_0$. So stated, the result is a Π_1^1 statement and it is not statable in first-order PA. However, Gentzen showed that any proof of a contradiction in first-order PA would imply the existence of a primitive recursive infinite descending chain of such ordinals. Hence the statement that no such chains exist is Π_2^0 . By Gödel's second incompleteness theorem, it is unprovable in first-order PA.⁸

Exactly the same device is employed by Kirby and Paris 1982 in their Theorem 2. They show that in their Hercules-hydra game every strategy is a winning strategy for Hercules (a Π_1^1 statement not statable in first-order PA, although statable if a function letter is added). The idea is that an ordinal $<\varepsilon_0$ can be assigned to every tree (hydra) in the game, and that after each play the ordinal is reduced. To get an independence result for first-order PA, they proved that the Π_3^0 statement that every recursive strategy is a winning strategy is unprovable in first-order PA. Gentzen himself proved the independence of the well-foundedness of *primitive recursive* descending chains of ordinals $<\varepsilon_0$ (in extended Cantor normal form). Therefore, it seems to the present writer that their independence result concerning the 'hydra' game can be improved to the independence of 'every primitive recursive strategy is a winning strategy' (for Hercules),

⁸ Actually, Gentzen eventually showed that the independence of ε_0 -induction could be shown directly, without recourse to Gödel's second incompleteness theorem (*Gentzen 1943*). The same is true of the Paris-Harrington theorem even though it is not presented that way in the *Handbook* proof.

rather than general recursive. This can be written as a Π_2^0 statement, improving their independence result for a Π_3^0 statement.

However, when Kirby and Paris say that they present ‘perhaps the first [independence result in first-order PA] which is, in an informal sense, purely number-theoretic in character (as opposed to metamathematical or combinatorial)’ (1982, p. 285) what they say ignores Gentzen’s statement (the basis of their own paper!), which is neither metamathematical nor combinatorial.

However, the Gentzen statement is not, as *he himself formulated it*, number-theoretic in character. Gentzen’s original statement, the unprovability of the non-existence of primitive recursive descending sequences of ordinals in extended Cantor normal form $<\varepsilon_0$ in first-order PA, though proving the independence of a purely ‘mathematical’ statement in first-order PA, does require coding, since first-order PA does not explicitly talk about ordinals and their representations in extended Cantor normal form. Goodstein’s reformulation, however, achieves a purely arithmetical, or number-theoretic formulation, without any coding of statements about ordinals in first-order PA.

Indeed, as is universally agreed, the Kirby-Paris result on the unprovability of the termination of Goodstein sequences, even of a special kind, is a mathematical, purely number-theoretic, result unprovable in first-order PA. However, as I said, the community appears to have read *Kirby and Paris 1982* a bit carelessly and largely ignored *Goodstein 1944*. Already on the first page, Kirby and Paris rightly say, ‘the first result of our paper is an improvement of a theorem of Goodstein’ (1982, p. 285). Remembering that their first result is not the *truth* of the statement that the Goodstein sequences terminate, but

rather its *independence* in first-order PA, one should understand the right situation.⁹ Goodstein already proved a related independence result, weaker than that proved by Kirby and Paris (because it proves the independence of an even stronger statement). In fact, Kirby and Paris explicitly state two facts: first, that Gentzen ‘showed that using transfinite induction on ordinals below ε_0 one can prove the consistency of P’ (1982, p. 287); second, that ‘Goodstein proved the following: if $h: N \rightarrow N$ is a non-decreasing function, define an h -Goodstein sequence b_0, b_1, \dots by letting b_{i+1} be the result of replacing every $h(i)$ in the base $h(i)$ representation of b_i by $h(i+1)$, and subtracting 1. Then the statement “for every non-decreasing h , every h -Goodstein sequence eventually reaches 0” is equivalent to transfinite induction below ε_0 ’ (1982, p. 287). So stated, the Goodstein statement is even stronger than the one usually quoted, and therefore even more surprising.

As Goodstein states it, the statement is Π_1^1 and therefore not statable in first-order PA. In fact, since Gentzen’s argument actually showed that the non-existence of *primitive*

⁹ Misunderstanding Kirby and Paris’s first theorem, Henle states that the *truth* of the quoted result on ‘Goodstein sequences’ is an improvement, due to *Kirby and Paris 1982* of Goodstein’s original result (see *Henle 1986*, p. 47). Actually, of course, Goodstein’s original statement was *stronger* than the one shown to be unprovable in first-order PA by Kirby and Paris, where the superbases must be increased by one. (*Rathjen 2015* calls this the ‘shift’ function.) Since the superbases can go up even faster without this restriction, Goodstein’s sequences of this kind get enormously large even more quickly, making the statement that they must eventually hit 0 even more surprising. As to the relation of Goodstein’s statement to ε_0 -induction, see the remarks on section 3 on von Plato’s account quoted in *Rathjen 2015* of the correspondence between Bernays and Goodstein.

recursive descending sequences of ordinals $<\varepsilon_0$ implies the consistency of first-order PA, and therefore is unprovable in first-order PA, equivalently the statement that primitive recursive Goodstein sequences must eventually reach 0 is unprovable in first-order PA. This last is a Π_2^0 statement, statable in first-order PA.

It is unclear to me whether Goodstein was aware of this latter fact. Probably he was not. All that I can find in his paper that is relevant is the vague assertion about decreasing sequences of ordinals (in extended Cantor normal form): ‘For every constructively given sequence of ordinals the sequence m_r is general recursive though not perhaps primitive recursive in every case’ (1944, p. 34).

Various remarks ought to be made about Goodstein’s paper.¹⁰ First, Goodstein himself expresses no recognition of how very surprising his result is. This is emphasized in the paper by Kirby and Paris, even for the weaker form that is the focus of their paper, where ‘superbases’ are raised by 1. (Henle, as quoted above, also emphasizes it.)¹¹ Second, almost all articles on the subject,¹² including the Wikipedia entry on ‘Goodstein’s

¹⁰ One referee has asked me to explain whether these remarks are or are not relevant to the main thesis of the present paper. I will add such explanations in footnotes.

¹¹ Obviously, this is not directly relevant to the main point of this paper. However, it gives some explanation for the relative neglect of Goodstein’s paper until its revival by Kirby and Paris. Whether Goodstein himself was aware of the surprising character of his result is yet another puzzle.

¹² The main exceptions of this generalization known to me are *Kirby and Paris 1982* and *Rathjen 2015*. I have seen other papers that assume that Goodstein’s work on the subject was on what Rathjen calls the ‘shift’ function.

theorem’ and the entry on the same subject in Wolfram Math World, state that Goodstein sequences are those represented in Theorem 1 by *Kirby and Paris 1982* (i.e. on the ‘shift’ function. See footnote 10). In the Wikipedia entry on Goodstein’s theorem, Goodstein’s *original* more general notion is called ‘extended Goodstein sequences’.

A further point is the following. In the published paper, Goodstein puts little emphasis on the fact that he has found a number-theoretic equivalent of Gentzen’s incompleteness result. Rather, the emphasis is on the supposed ‘finitist’ character of his proof, at least of special cases, say up to ω^{ω} . What happened here is at least partially explained by the archival research by Jan von Plato, restated below in section 3.¹³

One must also mention a few other things. Gentzen’s techniques can be developed not only to show the simple consistency but even the 1-consistency (Σ_1 -correctness) of first-order PA. Also, the Gentzen proof (even of 1-consistency) can be extended to the case where a one place function letter is added. (Given pairing functions, etc., this is equivalent to the system with infinitely many function letters of any number of variables.) On this basis, Goodstein’s formulation with arbitrary Goodstein sequences could be stated and proved independent. Moreover, on this basis, the Kreisel results characterizing the Π_2^0 statements provable in first-order PA (in terms of ordinal recursion $<\varepsilon_0$) and the Π_1^1 form (or ‘no counterexample’) interpretation of arbitrary statements provable in first-order PA can be proved (*Kreisel 1951* and *1952*). (The latter was already known to Gödel

As I have already stated, this fact makes it clear that Goodstein’s article did not generate a long search for a proof of his result in first-order PA. But that in addition, very few writers on the subject have even glanced at Goodstein’s original paper.

¹³ This, a considerable change in emphasis from the original paper, is another reason why Goodstein’s credit for an independence result was neglected until *Rathjen 2015*.

much earlier but was not published. See the introductory note to *Gödel 1938*, section 7, pp. 82-84.)¹⁴

Finally, one should also mention that the paper by *Ketonen and Solovay 1981* on the Paris-Harrington theorem bases everything on Gentzen's work on the unprovability of ε_0 -induction, and the corollaries by Kreisel (and Gödel) on which functions are provably recursive in first-order PA. The same is also true not only of *Kirby and Paris 1982*, but also of the shorter proof of their result by *Cichon 1983*.¹⁵

3. A Final Section

This was the situation in the early years. However, already in 2010, Craig Smoryński, reviewing a book by Peter Smith, stresses that the Paris-Harrington statement was *not* the first mathematical statement proved independent of first-order PA, contrary to the impression many people appear to have (*Smoryński 2010, Smith 2007*). Rather, he mentions Gentzen and the applications of his work by Kreisel. (Also, he observes that the Paris-Harrington result is an improvement of earlier work of Paris.) Even more important, Smith himself on his *Logic Matters* blog, remarking on Smoryński's review, says that several people have warned him similarly about Gentzen, and that he is embarrassed that

¹⁴ Goodstein's original formulation would lead to an independence result only for this (conservative) extension of first-order PA. See the discussion of this in *Rathjen 2015*. The other remarks (Kreisel, Gödel, etc.) show the fertility of the results on this extension.

¹⁵ The point here is that, not only Kirby and Paris, but even the work by Ketunen and Solovay on the Paris-Harrington theorem, are based on Gentzen's work. So why was *Gentzen 1936* and *1943* ignored in work devoted to 'mathematical incompleteness in Peano arithmetic'?

he did not take it out of later versions of his book. (Probably it has been taken out of the second edition.)

Most important, *Rathjen 2015* published a highly technically precise and informative treatment of Goodstein's original paper, including an interesting recovery of Bernays's correspondence with Goodstein, due to Jan von Plato. Bernays was chosen as the referee of Goodstein's paper. He pointed out that the termination of arbitrary Goodstein sequences cannot be formulated in first-order PA (it is Π_1^1) and that one needs to add a function letter, and then the Gentzen independence proof would still apply. Unfortunately, this led Goodstein to delete the explicit claim of an independence result and concentrate on the supposed 'finitist' character of his proof, at least up to certain ordinals $< \varepsilon_0$, for example ω^{ω^ω} . Bernays had cautioned him about this also, but in this case he ignored his advice.¹⁶ He could have found an independence result in first-order PA itself using primitive recursive sequences; but as we saw above, he doesn't seem to be aware of this. Goodstein, as mentioned above, does vaguely talk of general recursive sequences. If he thought that this would be independent in first-order PA, he could have stated an independence result in first-order PA, although it would be Π_3^0 .¹⁷

¹⁶ This account of von Plato's is taken nearly verbatim from *Rathjen 2015*, though I talk about one function letter, where Bernays talks about adding arbitrary function variables. These are equivalent as sketched above.

¹⁷ *Rathjen 2015* says that Goodstein 'could have followed Bernays's suggestion or he could have found an independence result for PA proper by scrutinizing Gentzen's proof which only utilizes the termination of primitive recursive sequences of ordinals'. However, he adds in footnote 2, 'this statement may be a bit too strong since it assumes that Goodstein had penetrated the details of Gentzen's rather difficult paper [*Gentzen 1936*]'

Rathjen gives a very careful treatment of the work of Goodstein, more strictly done than in the present paper or Goodstein's original. He also gives a treatment of what he calls 'special Goodstein sequences' using only tools that would have been available in the 1940's and 1950's. These are those where the 'superbase' increases by one, and it is these that are treated by *Kirby and Paris 1982*.¹⁸ As we have seen, even today (2021) the Wikipedia entry on Goodstein's theorem states that Goodstein's sequences are only these and calls Goodstein's actual sequences 'extended Goodstein sequences'. The surprising character of Goodstein's result is elaborated. (We have also seen that the Wolfram Math World entry also states that Goodstein sequences are those where the superbase is increased by one.) However, *this* Wikipedia entry is aware of Gentzen's very early priority over Paris and Harrington for a mathematical independence proof but does not appear to give Goodstein such a priority.¹⁹

Rathjen, too, even in his abstract, shares my doubts as to whether there was really a long search 'for strictly mathematical examples of an incompleteness in **PA** really

¹⁸ In footnote 4, p. 231, *Rathjen 2015* says that 'around 1979, Diana Schmidt proved that Kruskal's theorem elementarily implies that the ordinal representation system for Γ_0 is well-founded [Schmidt! See *Naming and Necessity*, pp. 83-84]. She even wrote (p. 61) that she didn't know of any applications of her result to proof theory. This is quite surprising since in conjunction with proof-theoretic work of Feferman and Schütte from the 1960's it immediately implies the nowadays celebrated result that Kruskal's theorem is unprovable in predicative mathematics'.

¹⁹ Although it does say that Goodstein's statement is an early example of a statement demonstrably independent of first-order PA, it attributes everything to *Kirby and Paris 1982*.

Even now, if someone Googles 'Mathematical Incompleteness in Peano Arithmetic,' she is referred to a Wikipedia entry on the Paris-Harrington Theorem.

attained its “holy grail” status before the late 1970s’ (see also footnote 2). Although he says that ‘almost no direct moral is ever given’, he concludes his abstract by saying, ‘However, in relation to independence results, we think that both Gentzen and Goodstein are deserving of more credit’. I certainly think the statement about Gentzen and Goodstein as deserving of more credit is indeed a moral of his work. It accords, in any case, with the view that I have taken in the present paper.²⁰

²⁰ I would like to thank Harold Teichman and Yale Weiss for their editorial help. Special thanks go to Romina Padró for her invaluable help in producing the present and the original 2007 version of this paper. This paper has been completed with support from the Saul Kripke Center at the City University of New York, Graduate Center.

References

- Barwise, J. (ed.). 1977. *Handbook of Mathematical Logic*, Amsterdam: North-Holland.
- Buss, S. (ed.). 1998. *Handbook of Proof Theory*, Amsterdam: Elsevier.
- Cichon, E. A. 1983. 'A short proof of two recently discovered independence results using recursion theoretic methods', *Proceedings of the American Mathematical Society*, **87** (4), 704–6.
- Davis, M., Putnam, H., and Robinson, J. 1961. 'The Decision Problem for Exponential Diophantine Equations', *Annals of Mathematics*, **74** (3), 425-36.
- Feferman, S., Dawson Jr., J.W., Kleene, S. C., Moore, G. H., Solovay, R. M., and van Heijenoort, J. (eds.). 1986. *Kurt Gödel. Collected Works, Volume I. Publications 1929-1936*, New York: Oxford University Press.
- Feferman, S., Dawson Jr., J.W., Goldfarb, W., Parsons, C., Solovay, R. M. (eds.). 1995. *Kurt Gödel. Collected Works, Volume III. Unpublished Essays and Lectures*, New York: Oxford University Press.
- Gentzen, G. 1936. 'Die Widerspruchsfreiheit der reinen Zahlentheorie', *Mathematische Annalen*, **112**, 493-565.
- Gentzen, G. 1943. 'Beweisbarkeit und Unbeweisbarkeit von Anfangsfallen der Transfiniten Induktion in der reinen Zahlentheorie', *Mathematische Annalen*, **119**, 140-61.
- Gödel, K. 1931. 'Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I', *Monatshefte für Mathematik und Physik*, **38**, 173–98. Translated in *Feferman et. al. 1986* as 'On formally undecidable propositions of Principia Mathematica and related systems I', pp. 144-95.
- Gödel, K. 1934. 'On undecidable propositions of formal mathematical systems'. Lecture notes by Kleene, S.C., Rosser, J.B. Princeton University, Princeton. Reprinted in *Feferman et. al. 1986*, pp. 338-71.
- Gödel, K. 1938. 'Lecture at Zilsel's', in *Feferman et. al. 1995*, pp. 87-113.
- Goodstein, R. L. 1944. 'On the Restricted Ordinal Theorem', *Journal of Symbolic Logic*, **9** (2), 33-41.
- Graham, R. L., Rothschild, B. L, and Spencer, J. H. 1990. *Ramsey Theory*, 2nd Edition, New York: Wiley.
- Henle, J. 1986. *An Outline of Set Theory. Problem books in mathematics*, Berlin: Springer-Verlag.

- Ketonen, J., and Solovay, R. 1981. ‘Rapidly Growing Ramsey Functions’, *Annals of Mathematics*, 113(2), second series, 267-314.
- Kirby, L. and Paris, J. 1982. ‘Accessible Independence Results for Peano Arithmetic’, *Bulletin of the London Mathematical Society*, 14 (4), 285-93.
- Kreisel, G. 1951. ‘On the interpretation of non-finitist proofs I’, *Journal of Symbolic Logic*, 16 (4), 241-67.
- Kreisel, G. 1952. ‘On the interpretation of non-finitist proofs II’, *Journal of Symbolic Logic*, 17 (1), 43-58.
- Kripke, S. A. 1972/80. *Naming and Necessity*. Oxford: Basil Blackwell and Cambridge, Massachusetts: Harvard University Press. First published in D. Davidson and G. Harman (eds.), *Semantics of Natural Language* (2nd ed.), Dordrecht: D. Reidel Publishing Company, pp. 253-355; Addenda, pp. 763-9.
- Matiyasevich, Y. V. 1970. ‘Enumerable sets are Diophantine’, *Doklady Akademii Nauk SSSR* (in Russian), 191, 279–82. English translation in *Soviet Mathematics* 11 (2): 354–7.
- Paris, J. and Harrington, L. 1977. ‘A Mathematical Incompleteness in Peano Arithmetic’, in *Barwise 1977*, pp. 1133–42.
- Rathjen, M. 2015. ‘Goodstein’s Theorem Revisited’, in R. Kahle and M. Rathjen (eds.), *Gentzen’s Centenary: The Quest for Consistency*, New York: Springer, pp. 229-42.
- Smith, P. 2007. *An Introduction to Gödel’s Theorems*, Cambridge: Cambridge University Press.
- Smoryński, C. 2010. ‘Review of P. Smith, *An introduction to Gödel’s theorems*’, *Philosophia Mathematica*, 18 (1), 122-7.
- Weisstein, E. W. Undated. ‘Goodstein’s Theorem’, in *MathWorld*—a Wolfram Web Resource, retrieved 28 Sep. 20, from <https://mathworld.wolfram.com/GoodsteinsTheorem.html>.
- Wikipedia contributors. 2021 (June 8). ‘Goodstein’s theorem’, in *Wikipedia, The Free Encyclopedia*, retrieved June 8, from https://en.wikipedia.org/w/index.php?title=Goodstein%27s_theorem&oldid=971773400.